

NCERT Solutions Class 11 Maths Chapter 3 Trigonometric Functions

Question 1:

Find the radian measures corresponding to the following degree measures:

- (i) 25° (ii) $-47^\circ 30'$ (iii) 240° (iv) 520°

Solution:

We know that $180^\circ = \pi$ radian

Therefore,

$$\begin{aligned} \text{(i)} \quad 25^\circ &= \frac{\pi}{180} \times 25 \text{ radian} \\ &= \frac{5\pi}{36} \text{ radian} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad -47^\circ 30' &= -\left(47\frac{1}{2}\right)^\circ = -\left(\frac{95}{2}\right)^\circ \\ &= -\left(\frac{95}{2}\right)^\circ = \frac{\pi}{180} \times -\left(\frac{95}{2}\right) \text{ radian} \\ &= -\frac{19\pi}{72} \text{ radian} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad 245^\circ &= \frac{\pi}{180} \times 240 \text{ radian} \\ &= \frac{3\pi}{4} \text{ radian} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad 520^\circ &= \frac{\pi}{180} \times 520 \text{ radian} \\ &= \frac{26\pi}{9} \text{ radian} \end{aligned}$$

Question 2:

Find the degree measures corresponding to the following radian measures $\left(\text{Use } \pi = \frac{22}{7}\right)$.

(i) $\frac{11}{16}$

(ii) -4

(iii) $\frac{5\pi}{3}$

(iv) $\frac{7\pi}{6}$

Solution:

We know that $180^\circ = \pi$ radian

Therefore,

$$\begin{aligned} \text{(i)} \quad \frac{11}{16} \text{ radian} &= \frac{180^\circ}{\pi} \times \frac{11}{16} \\ &= 180 \times \frac{7}{22} \times \frac{11}{16} \text{ deg} \\ &= \frac{315}{8} \text{ deg} \\ &= 39\frac{3}{8} \text{ deg} \\ &= 39 \text{ deg} + \frac{3}{8} \times 60 \text{ min} && [\because 1^\circ = 60'] \\ &= 39 \text{ deg} + 22\frac{1}{2} \text{ min} \\ &= 39 \text{ deg} + 22 \text{ min} + \frac{60}{2} \text{ sec} && [\because 1' = 60''] \\ &= 39^\circ 22' 30'' \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad -4 \text{ radian} &= \frac{180^\circ}{\pi} \times (-4) \\ &= 180 \times \frac{7}{22} \times (-4) \text{ deg} \\ &= -\frac{2520}{11} \text{ deg} \\ &= -229\frac{1}{11} \text{ deg} \\ &= -229 \text{ deg} + \frac{60}{11} \text{ min} && [\because 1^\circ = 60'] \\ &= -22 \text{ deg} + 5\frac{5}{11} \text{ min} \\ &= -22 \text{ deg} + 5 \text{ min} + \frac{5}{11} \times 60 \text{ sec} && [\because 1' = 60''] \\ &= -22^\circ 5' 27'' \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \frac{5\pi}{3} \text{ radian} &= \frac{180^\circ}{\pi} \times \frac{5\pi}{3} \\ &= 300^\circ \end{aligned}$$

$$\text{(iv)} \quad \frac{7\pi}{6} \text{ radian} = \frac{180^\circ}{\pi} \times \frac{7\pi}{6}$$

$$= 210^\circ$$

Question 3:

A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Solution:

A wheel makes 360 revolutions in 1 minute (60 seconds)

Therefore,

Number of revolutions made by the wheel in 1 second $= \frac{360}{60} = 6$

In one complete revolution, the wheel turns an angle of 2π radians

Hence, in 6 complete revolutions, it will turn an angle of $6 \times 2\pi = 12\pi$ radians

Thus, in one second, the wheel turns an angle of 12π radians.

Question 4:

Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by

an arc of length 22 cm. $\left(\text{Use } \pi = \frac{22}{7} \right)$

Solution:

As we know that if in a circle of radius r , an arc of length l subtends an angle of θ radians,

Then $l = r\theta$

Therefore, $\theta = \frac{l}{r}$ radian

$$\theta = \frac{22\text{cm}}{100\text{cm}} \text{ radian}$$

$$\theta = \frac{11}{50} \times \frac{180}{\pi} \text{ deg}$$

$$= \frac{11}{50} \times 180 \times \frac{7}{22} \text{ deg}$$

$$= \frac{63}{5} \text{ deg}$$

$$= 12\frac{3}{5} \text{ deg}$$

$$= 12 \text{ deg} + \frac{3}{5} \times 60 \text{ min}$$

$$= 12^\circ 36'$$

$$[\because 1^\circ = 60']$$

Thus, the required angle is $12^{\circ}36'$

Question 5:

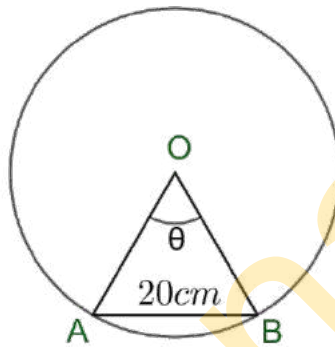
In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Solution:

Diameter of the circle = 40 cm

Therefore, Radius of the circle $r = \frac{40\text{cm}}{2} = 20\text{cm}$

Let AB be a chord of length 20 cm of the circle.



In $\triangle AOB$

$$AB = 20\text{cm}$$

$$OA = OB = r = 20\text{cm}$$

Hence, $\triangle AOB$ is an equilateral triangle

Thus, $\theta = 60^{\circ}$ or $\theta = \frac{\pi}{3}$ radian

As we know that if in a circle of radius r , an arc of length l subtends an angle of θ radians,

Then $l = r\theta$

Therefore,

$$\begin{aligned} l &= r\theta \\ AB &= 20\text{cm} \times \frac{\pi}{3} \\ &= \frac{20\pi}{3}\text{cm} \end{aligned}$$

Hence, the length of the minor arc of the chord is $\frac{20\pi}{3}\text{cm}$.

Question 6:

If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Solution:

Let the radii of the two circles be r and R .

Let an arc of length l subtend an angle of 60° at the centre of the circle of radius r , and 75° at the centre of the circle of radius R .

Now,

$$60^\circ = \frac{\pi}{3} \text{ radian and } 75^\circ = \frac{5\pi}{12} \text{ radian}$$

As we know that if in a circle of radius r , an arc of length l subtends an angle of θ radians,

Then $l = r\theta$

Therefore,

$$\begin{aligned} l &= r \times \frac{\pi}{3} & l &= R \times \frac{5\pi}{12} \\ &= \frac{\pi r}{3} & &= \frac{5\pi R}{12} \end{aligned} \quad \text{and}$$

Thus,

$$\begin{aligned} \frac{\pi r}{3} &= \frac{5\pi R}{12} \\ \frac{r}{R} &= \frac{5}{4} \end{aligned}$$

$$r : R = 5 : 4$$

Hence, the ratio of their radii is 5 : 4.

Question 7:

Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm

(ii) 15 cm

(iii) 21 cm

Solution:

As we know that if in a circle of radius r , an arc of length l subtends an angle of θ radians,

Then $l = r\theta$

(i) Radius, $r = 75\text{cm}$ and length of the arc, $l = 10\text{cm}$

$$\begin{aligned} \theta &= \frac{l}{r} \\ &= \frac{10\text{cm}}{75\text{cm}} \\ &= \frac{2}{15} \end{aligned}$$

Thus, $\theta = \frac{2}{15}$ radian

- (ii) Radius, $r = 75\text{cm}$ and length of the arc, $l = 15\text{cm}$

$$\begin{aligned}\theta &= \frac{l}{r} \\ &= \frac{15\text{cm}}{75\text{cm}} \\ &= \frac{1}{5}\end{aligned}$$

Thus, $\theta = \frac{1}{5}$ radian

- (iii) Radius, $r = 75\text{cm}$ and length of the arc, $l = 21\text{cm}$

$$\begin{aligned}\theta &= \frac{l}{r} \\ &= \frac{21\text{cm}}{75\text{cm}} \\ &= \frac{7}{25}\end{aligned}$$

Thus, $\theta = \frac{7}{25}$ radian

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

Question 1:

$\cos x = -\frac{1}{2}$, x lies in third quadrant.

Solution:

As we know that

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ &= \frac{1}{\left(-\frac{1}{2}\right)} \\ &= -2\end{aligned}$$

Now, $\sin^2 x + \cos^2 x = 1$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin x = \pm \sqrt{1 - \cos^2 x}$$

$$= \pm \sqrt{1 - \left(-\frac{1}{2}\right)^2}$$

$$= \pm \sqrt{1 - \frac{1}{4}}$$

$$= \pm \sqrt{\frac{3}{4}}$$

$$= \pm \frac{\sqrt{3}}{2}$$

Since, x lies in third quadrant, the value of $\sin x$ will be negative.
Therefore,

$$\sin x = -\frac{\sqrt{3}}{2}$$

Now,

$$\begin{aligned}\operatorname{cosec} x &= \frac{1}{\sin x} \\ &= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)} \\ &= -\frac{2}{\sqrt{3}}\end{aligned}$$

Now,

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} \\ &= \frac{\left(-\frac{1}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{3}\end{aligned}$$

Now,

$$\begin{aligned}\cot x &= \frac{1}{\tan x} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

Hence, $\sin x = -\frac{\sqrt{3}}{2}$, $\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$, $\sec x = -2$, $\tan x = \sqrt{3}$, and $\cot x = \frac{1}{\sqrt{3}}$

Question 2:

$\sin x = \frac{3}{5}$, x lies in second quadrant.

Solution:

As we know that

$$\begin{aligned}\operatorname{cosec} x &= \frac{1}{\sin x} \\ &= \frac{1}{\left(\frac{3}{5}\right)} \\ &= \frac{5}{3}\end{aligned}$$

Now, $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned}
 \cos^2 x &= 1 - \sin^2 x \\
 \cos x &= \pm \sqrt{1 - \sin^2 x} \\
 &= \pm \sqrt{1 - \left(\frac{3}{5}\right)^2} \\
 &= \pm \sqrt{1 - \frac{9}{25}} \\
 &= \pm \sqrt{\frac{16}{25}} \\
 &= \pm \frac{4}{5}
 \end{aligned}$$

Since, x lies in second quadrant, the value of $\cos x$ will be negative.
Therefore,

$$\cos x = -\frac{4}{5}$$

Now,

$$\begin{aligned}
 \sec x &= \frac{1}{\cos x} \\
 &= \frac{1}{\left(-\frac{4}{5}\right)} \\
 &= -\frac{5}{4}
 \end{aligned}$$

Now,

$$\begin{aligned}
 \tan x &= \frac{\sin x}{\cos x} \\
 &= \frac{\left(\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)} \\
 &= -\frac{3}{4}
 \end{aligned}$$

Now,

$$\begin{aligned}
 \cot x &= \frac{1}{\tan x} \\
 &= \frac{1}{\left(-\frac{3}{4}\right)} \\
 &= -\frac{4}{3}
 \end{aligned}$$

Hence, $\operatorname{cosec} x = \frac{5}{3}$, $\cos x = -\frac{4}{5}$, $\sec x = -\frac{5}{4}$, $\tan x = -\frac{3}{4}$, and $\cot x = -\frac{4}{3}$

Question 3:

$\cot x = \frac{3}{4}$, x lies in third quadrant.

Solution:

As we know that

$$\begin{aligned}\tan x &= \frac{1}{\cot x} \\ &= \frac{1}{\left(\frac{3}{4}\right)} \\ &= \frac{4}{3}\end{aligned}$$

Now, $1 + \tan^2 x = \sec^2 x$

$$\begin{aligned}\sec x &= \pm \sqrt{1 + \tan^2 x} \\ &= \pm \sqrt{1 + \left(\frac{4}{3}\right)^2} \\ &= \pm \sqrt{1 + \frac{16}{9}} \\ &= \pm \sqrt{\frac{25}{9}} \\ &= \pm \frac{5}{3}\end{aligned}$$

Since, x lies in third quadrant, the value of $\sec x$ will be negative.
Therefore,

$$\sec x = -\frac{5}{3}$$

Now,

$$\begin{aligned}\cos x &= \frac{1}{\sec x} \\ &= \frac{1}{\left(-\frac{5}{3}\right)} \\ &= -\frac{3}{5}\end{aligned}$$

Now,

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} \\ \sin x &= \tan x \cos x \\ &= \left(\frac{4}{3}\right) \times \left(-\frac{3}{5}\right) \\ &= -\frac{4}{5}\end{aligned}$$

Now,

$$\begin{aligned}\operatorname{cosec} x &= \frac{1}{\sin x} \\ &= \frac{1}{\left(-\frac{4}{5}\right)} \\ &= -\frac{5}{4}\end{aligned}$$

Hence, $\sin x = -\frac{4}{5}$, $\operatorname{cosec} x = -\frac{5}{4}$, $\cos x = -\frac{3}{5}$, $\sec x = -\frac{5}{3}$, and $\tan x = \frac{4}{3}$

Question 4:

$\sec x = \frac{13}{5}$, x lies in fourth quadrant.

Solution:

As we know that

$$\begin{aligned}\cos x &= \frac{1}{\sec x} \\ &= \frac{1}{\left(\frac{13}{5}\right)} \\ &= \frac{5}{13}\end{aligned}$$

Now, $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned}
 \sin^2 x &= 1 - \cos^2 x \\
 \sin x &= \pm \sqrt{1 - \cos^2 x} \\
 &= \pm \sqrt{1 - \left(\frac{5}{13}\right)^2} \\
 &= \pm \sqrt{1 - \frac{25}{169}} \\
 &= \pm \sqrt{\frac{144}{169}} \\
 &= \pm \frac{12}{13}
 \end{aligned}$$

Since, x lies in fourth quadrant, the value of $\sin x$ will be negative.

Therefore,

$$\sin x = -\frac{12}{13}$$

Now,

$$\begin{aligned}
 \operatorname{cosec} x &= \frac{1}{\sin x} \\
 &= \frac{1}{\left(-\frac{12}{13}\right)} \\
 &= -\frac{13}{12}
 \end{aligned}$$

Now,

$$\begin{aligned}
 \tan x &= \frac{\sin x}{\cos x} \\
 &= \frac{\left(-\frac{12}{13}\right)}{\left(\frac{5}{13}\right)} \\
 &= -\frac{12}{5}
 \end{aligned}$$

Now,

$$\begin{aligned}\cot x &= \frac{1}{\tan x} \\ &= \frac{1}{\left(-\frac{12}{5}\right)} \\ &= -\frac{5}{12}\end{aligned}$$

Hence, $\sin x = -\frac{12}{13}$, $\operatorname{cosec} x = -\frac{13}{12}$, $\cos x = \frac{5}{13}$, $\tan x = -\frac{12}{5}$, and $\cot x = -\frac{5}{12}$

Question 5:

$\tan x = -\frac{5}{12}$, x lies in second quadrant.

Solution:

As we know that

$$\begin{aligned}\cot x &= \frac{1}{\tan x} \\ &= \frac{1}{\left(-\frac{5}{12}\right)} \\ &= -\frac{12}{5}\end{aligned}$$

Now, $1 + \tan^2 x = \sec^2 x$

$$\begin{aligned}\sec x &= \pm \sqrt{1 + \tan^2 x} \\ &= \pm \sqrt{1 + \left(-\frac{5}{12}\right)^2} \\ &= \pm \sqrt{1 + \frac{25}{144}} \\ &= \pm \sqrt{\frac{169}{144}} \\ &= \pm \frac{13}{12}\end{aligned}$$

Since, x lies in second quadrant, the value of $\sec x$ will be negative.
Therefore,

$$\sec x = -\frac{13}{12}$$

Now,

$$\begin{aligned}\cos x &= \frac{1}{\sec x} \\ &= \frac{1}{\left(-\frac{13}{12}\right)} \\ &= -\frac{12}{13}\end{aligned}$$

Now,

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} \\ \sin x &= \tan x \cos x \\ &= \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right) \\ &= \frac{5}{13}\end{aligned}$$

Now,

$$\begin{aligned}\operatorname{cosec} x &= \frac{1}{\sin x} \\ &= \frac{1}{\left(\frac{5}{13}\right)} \\ &= \frac{13}{5}\end{aligned}$$

Hence, $\sin x = \frac{5}{13}$, $\operatorname{cosec} x = \frac{13}{5}$, $\cos x = -\frac{12}{13}$, $\sec x = -\frac{13}{12}$, and $\cot x = -\frac{12}{5}$

Find the values of the trigonometric functions in Exercises 6 to 10.

Question 6:

$\sin 765^\circ$

Solution:

It is known that the value of $\sin x$ repeat after an interval of 2π or 360° .

Therefore,

$$\begin{aligned}\sin 765^\circ &= \sin(2 \times 360^\circ + 45^\circ) \\ &= \sin 45^\circ \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Question 7:

$$\operatorname{cosec}(-1410^\circ)$$

Solution:

It is known that the value of $\sin x$ repeat after an interval of $2n$ or 360° .
Therefore,

$$\begin{aligned}\operatorname{cosec}(-1410^\circ) &= \operatorname{cosec}(4 \times 360^\circ - 1410^\circ) \\ &= \operatorname{cosec}(1440^\circ - 1410^\circ) \\ &= \operatorname{cosec}30^\circ \\ &= 2\end{aligned}$$

Question 8:

$$\tan \frac{19\pi}{3}$$

Solution:

It is known that the value of $\tan x$ repeat after an interval of n or 180° .
Therefore,

$$\begin{aligned}\tan \frac{19\pi}{3} &= \tan 6\frac{1}{3}\pi \\ &= \tan \left(6\pi + \frac{1}{3}\pi \right) \\ &= \tan \frac{\pi}{3} \\ &= \tan 60^\circ \\ &= \sqrt{3}\end{aligned}$$

Question 9:

$$\sin \left(-\frac{11\pi}{3} \right)$$

Solution:

It is known that the value of $\sin x$ repeat after an interval of $2n$ or 360° .

Therefore,

$$\begin{aligned}\sin\left(-\frac{11\pi}{3}\right) &= \sin\left(2 \times 2\pi - \frac{11\pi}{3}\right) \\ &= \sin\frac{\pi}{3} \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Question 10:

$$\cot\left(-\frac{15\pi}{4}\right)$$

Solution:

It is known that the value of $\cos x$ repeat after an interval of n or 180° .

Therefore,

$$\begin{aligned}\cot\left(-\frac{15\pi}{4}\right) &= \cot\left(4\pi - \frac{15\pi}{4}\right) \\ &= \cot\frac{\pi}{4} \\ &= \cot 45^\circ \\ &= 1\end{aligned}$$

EXERCISE 3.3

Prove that:

Question 1:

$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$$

Solution:

$$\begin{aligned} LHS &= \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2 \\ &= \frac{1}{4} + \frac{1}{4} - 1 \\ &= \frac{1+1-4}{4} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} = RHS \end{aligned}$$

Question 2:

$$2\sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$$

Solution:

$$\begin{aligned} LHS &= 2\sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} \\ &= 2\left(\frac{1}{2}\right)^2 + \operatorname{cosec}^2 \left(\pi + \frac{\pi}{6}\right) \times \left(\frac{1}{2}\right)^2 \\ &= 2 \times \frac{1}{4} + \left(-\operatorname{cosec} \frac{\pi}{6}\right)^2 \times \frac{1}{4} \\ &= \frac{1}{2} + (-2)^2 \times \frac{1}{4} \\ &= \frac{1}{2} + 1 \\ &= \frac{1+2}{2} \\ &= \frac{3}{2} = RHS \end{aligned}$$

Question 3:

$$\cot^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} = 6$$

Solution:

$$\begin{aligned} LHS &= \cot^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} \\ &= (\sqrt{3})^2 + \operatorname{cosec}^2 \left(\pi - \frac{\pi}{6} \right) + 3 \left(\frac{1}{\sqrt{3}} \right)^2 \\ &= 3 + \operatorname{cosec}^2 \frac{\pi}{6} + 3 \times \frac{1}{3} \\ &= 3 + 2 + 1 \\ &= 6 = RHS \end{aligned}$$

Question 4:

$$2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} = 10$$

Solution:

$$\begin{aligned} LHS &= 2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} \\ &= 2 \sin^2 \left(\pi - \frac{\pi}{4} \right) + 2 \times \left(\frac{1}{\sqrt{2}} \right)^2 + 2 \times (2)^2 \\ &= 2 \sin^2 \frac{\pi}{4} + 2 \times \frac{1}{2} + 2 \times 4 \\ &= 2 \times \left(\frac{1}{\sqrt{2}} \right)^2 + 1 + 8 \\ &= 1 + 9 \\ &= 10 = RHS \end{aligned}$$

Question 5:

Find the value of:

(i) $\sin 75^\circ$

(ii) $\tan 15^\circ$

Solution:

(i) $\sin 75^\circ$

$$\begin{aligned}\sin 75^\circ &= \sin (45^\circ + 30^\circ) \\&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \quad \left[\because \sin (x + y) = \sin x \cos y + \cos x \sin y \right] \\&= \left(\frac{1}{\sqrt{2}} \right) \times \left(\frac{\sqrt{3}}{2} \right) + \left(\frac{1}{\sqrt{2}} \right) \times \left(\frac{1}{2} \right) \\&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\&= \frac{\sqrt{3} + 1}{2\sqrt{2}}\end{aligned}$$

(ii) $\tan 15^\circ$

$$\begin{aligned}\tan 15^\circ &= \tan (45^\circ - 30^\circ) \\&= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \quad \left[\because \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \right] \\&= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} \\&= \frac{\frac{\sqrt{3} - 1}{\sqrt{3}}}{\frac{\sqrt{3} + 1}{\sqrt{3}}} \\&= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \\&= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \quad \text{[By rationalizing]} \\&= \frac{(\sqrt{3} - 1)^2}{3 - 1} \\&= \frac{3 + 1 - 2\sqrt{3}}{2} \\&= \frac{2(2 - \sqrt{3})}{2} \\&= 2 - \sqrt{3}\end{aligned}$$

Prove the following:

Question 6:

$$\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$$

Solution:

$$\begin{aligned} LHS &= \cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) \\ &= \frac{1}{2}\left[2\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)\right]+\frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)\right] \\ &= \left[\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}+\cos\left\{\left(\frac{\pi}{4}-x\right)-\left(\frac{\pi}{4}-y\right)\right\}\right]\right. \\ &\quad \left.+\frac{1}{2}\left[\cos\left\{\left(\frac{\pi}{4}-x\right)+\left(\frac{\pi}{4}-y\right)\right\}-\cos\left\{\left(\frac{\pi}{4}-x\right)-\left(\frac{\pi}{4}-y\right)\right\}\right]\right] \\ &\quad \left[\begin{array}{l} \because 2\cos A \cos B = \cos(A+B) + \cos(A-B) \\ -2\sin A \sin B = \cos(A+B) - \cos(A-B) \end{array}\right] \\ &= \sin(x+y) = RHS \\ &= \left[\frac{1}{2}\left[\cos\left\{\frac{\pi}{4}-x+\frac{\pi}{4}-y\right\}+\cos\left\{\frac{\pi}{4}-x-\frac{\pi}{4}+y\right\}\right]\right. \\ &\quad \left.+\frac{1}{2}\left[\cos\left\{\frac{\pi}{4}-x+\frac{\pi}{4}-y\right\}-\cos\left\{\frac{\pi}{4}-x-\frac{\pi}{4}+y\right\}\right]\right] \\ &= \frac{1}{2}\left[\cos\left\{\frac{\pi}{2}-(x+y)\right\}+\cos\{-(x-y)\}+\cos\left\{\frac{\pi}{2}-(x+y)\right\}-\cos\{-(x-y)\}\right] \\ &= \frac{1}{2}\left[2\cos\left\{\frac{\pi}{2}-(x+y)\right\}\right] \\ &= \sin(x+y) \quad \left[\because \cos\left(\frac{\pi}{2}-A\right)=\sin A\right] \\ &= RHS \end{aligned}$$

Question 7:

$$\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$$

Solution:

$$LHS = \frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$

$$= \frac{\left(\frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x}\right)}{\left(\frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}\right)}$$

$$= \frac{\left(\frac{1 + \tan x}{1 - \tan x}\right)}{\left(\frac{1 - \tan x}{1 + \tan x}\right)}$$

$$= \left(\frac{1 + \tan x}{1 - \tan x}\right) \times \left(\frac{1 + \tan x}{1 - \tan x}\right)$$

$$= \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$$

$$= RHS$$

$$\left[\begin{array}{l} \because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \& \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{array} \right]$$

Question 8:

$$\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \cot^2 x$$

Solution:

$$LHS = \frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}$$

$$= \frac{(-\cos x) \times (\cos x)}{(\sin x) \times (-\sin x)}$$

$$= \frac{-\cos^2 x}{-\sin^2 x}$$

$$= \left(\frac{\cos x}{\sin x}\right)^2$$

$$= \cot^2 x$$

$$= RHS$$

$$\left[\begin{array}{l} \because \cos(\pi+x) = -\cos x \\ \Rightarrow \cos(-x) = \cos x \\ \Rightarrow \cos\left(\frac{\pi}{2}+x\right) = -\sin x \\ \Rightarrow \sin(\pi-x) = \sin x \end{array} \right]$$

$$\left[\because \cot x = \frac{\cos x}{\sin x} \right]$$

Question 9:

$$\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$$

Solution:

$$LHS = \cos\left(\frac{3\pi}{2} + x\right) \cos(2\pi + x) \left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x) \right]$$

$$= \cos\left\{\pi + \left(\frac{\pi}{2} + x\right)\right\} \cos x \left[\cot\left\{\pi + \left(\frac{\pi}{2} - x\right)\right\} + \cot x \right]$$

$$= -\cos\left(\frac{\pi}{2} + x\right) \cos x \left[\cot\left(\frac{\pi}{2} - x\right) + \cot x \right]$$

$$= -(-\sin x) \cos x [\tan x + \cot x]$$

$$= \sin x \cos x \left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \right]$$

$$= \sin^2 x + \cos^2 x$$

$$= 1$$

$$= RHS$$

$$\left[\begin{array}{l} \because \cos(2n\pi + \theta) = \cos \theta \\ \Rightarrow \cot(2n\pi + \theta) = \cot \theta \end{array} \right]$$

$$\left[\begin{array}{l} \because \cos(\pi + \theta) = -\cos \theta \\ \Rightarrow \cot(\pi + \theta) = \cot \theta \end{array} \right]$$

$$\left[\begin{array}{l} \because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \\ \Rightarrow \cot(2n\pi + \theta) = \cos \theta \end{array} \right]$$

Question 10:

$$\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$$

Solution:

$$LHS = \sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x$$

$$= \cos(n+2)x \cdot \cos(n+1)x + \sin(n+2)x \cdot \sin(n+1)x$$

$$= \cos\{(n+2)x - (n+1)x\}$$

$$= \cos\{n+2 - n - 1\}x$$

$$= \cos x$$

$$= RHS$$

$$[\because \cos(A - B) = \cos A \cos B + \sin A \sin B]$$

Question 11:

$$\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$$

Solution:

$$\begin{aligned}LHS &= \cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) \\&= -2 \sin\left(\frac{\left(\frac{3\pi}{4} + x\right) + \left(\frac{3\pi}{4} - x\right)}{2}\right) \sin\left(\frac{\left(\frac{3\pi}{4} + x\right) - \left(\frac{3\pi}{4} - x\right)}{2}\right) \\&\quad \left[\because \cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)\right] \\&= -2 \sin\left(\frac{\frac{3\pi}{4} + x + \frac{3\pi}{4} - x}{2}\right) \sin\left(\frac{\frac{3\pi}{4} + x - \frac{3\pi}{4} + x}{2}\right) \\&= -2 \sin\left(\frac{3\pi}{4}\right) \sin\left(\frac{2x}{2}\right) \\&= -2 \sin\left(\pi - \frac{\pi}{4}\right) \sin x \\&= -2 \sin \frac{\pi}{4} \sin x \quad \left[\because \sin(\pi - \theta) = \sin \theta\right] \\&= -2 \times \frac{1}{\sqrt{2}} \times \sin x \\&= -\sqrt{2} \sin x \\&= RHS\end{aligned}$$

Question 12:

$$\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$$

Solution:

$$LHS = \sin^2 6x - \sin^2 4x$$

$$= (\sin 6x + \sin 4x)(\sin 6x - \sin 4x)$$

$$= \left[2\sin\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right) \right] \times \left[2\cos\left(\frac{6x+4x}{2}\right)\sin\left(\frac{6x-4x}{2}\right) \right]$$

$$\left[\begin{array}{l} \because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \& \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \end{array} \right]$$

$$= [2\sin 5x \cos x] \times [2\cos 5x \sin x]$$

$$= [2\sin 5x \cos 5x] \times [2\sin x \cos x]$$

$$= [\sin(5x+5x) + \sin(5x-5x)] \times [\sin(x+x) + \sin(x-x)]$$

$$[\because 2\sin A \cos B = \sin(A+B) + \sin(A-B)]$$

$$= [\sin 10x + \sin 0] \times [\sin 2x + \sin 0]$$

$$= [\sin 10x + 0] \times [\sin 2x + 0]$$

$$= \sin 2x \sin 10x$$

$$= RHS$$

Question 13:

$$\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$$

Solution:

$$LHS = \cos^2 2x - \cos^2 6x$$

$$= (\cos 2x + \cos 6x)(\cos 2x - \cos 6x) \quad \left[\because a^2 - b^2 = (a+b)(a-b) \right]$$

$$= \left[2 \cos \left(\frac{2x+6x}{2} \right) \cos \left(\frac{2x-6x}{2} \right) \right] \times \left[-2 \sin \left(\frac{2x+6x}{2} \right) \sin \left(\frac{2x-6x}{2} \right) \right]$$

$$\left[\begin{array}{l} \because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\ \& \cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \end{array} \right]$$

$$= [2 \cos 4x \cos (-2x)] \times [-2 \sin 4x \sin (-2x)]$$

$$= [2 \cos 4x \cos 2x] \times [-2 \sin 4x (-\sin 2x)] \quad \left[\begin{array}{l} \because \cos(-\theta) = \cos \theta \\ \& \sin(-\theta) = -\sin \theta \end{array} \right]$$

$$= [2 \cos 4x \cos 2x] \times [2 \sin 4x \sin 2x]$$

$$= [2 \cos 4x \sin 4x] \times [2 \cos 2x \sin 2x]$$

$$= [\sin(4x+4x) - \sin(4x-4x)] \times [\sin(2x+2x) - \sin(2x-2x)]$$

$$\left[\because 2 \cos A \sin B = \sin(A+B) - \sin(A-B) \right]$$

$$= [\sin 8x - \sin 0] \times [\sin 4x - \sin 0]$$

$$= [\sin 8x - 0] \times [\sin 4x - 0]$$

$$= \sin 4x \sin 8x$$

$$= RHS$$

Question 14:

$$\sin 2x + 2 \sin 4x + \sin 6x = 4 \cos^2 x \sin 4x$$

Solution:

$$\begin{aligned}
LHS &= \sin 2x + 2 \sin 4x + \sin 6x \\
&= [\sin 2x + \sin 6x] + 2 \sin 4x \\
&= \left[2 \sin \left(\frac{2x+6x}{2} \right) \cos \left(\frac{2x-6x}{2} \right) \right] + 2 \sin 4x & \left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= [2 \sin 4x \cos(-2x)] + 2 \sin 4x \\
&= 2 \sin 4x \cos 2x + 2 \sin 4x \\
&= 2 \sin 4x (\cos 2x + 1) \\
&= 2 \sin 4x (2 \cos^2 x - 1 + 1) & [\because \cos 2x = 2 \cos^2 x - 1] \\
&= 2 \sin 4x (2 \cos^2 x) \\
&= 4 \cos^2 x \sin 4x \\
&= RHS
\end{aligned}$$

Question 15:

$$\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$$

Solution:

$$\begin{aligned}
LHS &= \cot 4x (\sin 5x + \sin 3x) \\
&= \cot 4x \left[2 \sin \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) \right] & \left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= \frac{\cos 4x}{\sin 4x} [2 \sin 4x \cos x] & [\because \cos 2x = 2 \cos^2 x - 1] \\
&= 2 \cos 4x \cos x \\
&= 2 \cos 4x \cos x \times \frac{\sin x}{\sin x} \\
&= \frac{\cos x}{\sin x} \times [2 \cos 4x \sin x] \\
&= \cot x [\sin(4x+x) - \sin(4x-x)] & [\because 2 \cos A \sin B = \sin(A+B) - \sin(A-B)] \\
&= \cot x (\sin 5x - \sin 3x) \\
&= RHS
\end{aligned}$$

Question 16:

$$\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$$

Solution:

$$\begin{aligned} LHS &= \frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} \\ &= \frac{\left[-2 \sin \left(\frac{9x+5x}{2} \right) \sin \left(\frac{9x-5x}{2} \right) \right]}{\left[2 \cos \left(\frac{17x+3x}{2} \right) \sin \left(\frac{17x-3x}{2} \right) \right]} \\ &= \frac{[-2 \sin 7x \sin 2x]}{[2 \cos 10x \sin 7x]} \\ &= -\frac{\sin 2x}{\cos 10x} \\ &= RHS \end{aligned}$$

$$\begin{aligned} &\left[\because \cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right] \\ &\left[\& \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right] \end{aligned}$$

Question 17:

$$\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$$

Solution:

$$\begin{aligned} LHS &= \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} \\ &= \frac{\left[2 \sin \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) \right]}{\left[2 \cos \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) \right]} \\ &= \frac{\sin 4x}{\cos 4x} \\ &= \tan 4x \\ &= RHS \end{aligned}$$

$$\begin{aligned} &\left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\ &\left[\& \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \end{aligned}$$

Question 18:

$$\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{x-y}{2}$$

Solution:

$$LHS = \frac{\sin x - \sin y}{\cos x + \cos y}$$

$$= \frac{\left[2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \right]}{\left[2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \right]}$$

$$= \frac{\sin\left(\frac{x-y}{2}\right)}{\cos\left(\frac{x-y}{2}\right)}$$

$$= \tan \frac{x-y}{2}$$

$$= RHS$$

$$\left[\begin{array}{l} \because \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \\ \& \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \end{array} \right]$$

Question 19:

$$\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$$

Solution:

$$LHS = \frac{\sin x + \sin 3x}{\cos x + \cos 3x}$$

$$= \frac{\left[2 \sin\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right) \right]}{\left[2 \cos\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right) \right]}$$

$$= \frac{\sin 2x}{\cos 2x}$$

$$= \tan 2x$$

$$= RHS$$

$$\left[\begin{array}{l} \because \sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\ \& \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \end{array} \right]$$

Question 20:

$$\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x$$

Solution:

$$\begin{aligned} LHS &= \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} \\ &= \frac{\left[2 \cos \left(\frac{x+3x}{2} \right) \sin \left(\frac{x-3x}{2} \right) \right]}{-[\cos^2 x - \sin^2 x]} \\ &= \frac{2 \cos 2x \sin(-x)}{-\cos 2x} \\ &= 2 \sin x \\ &= RHS \end{aligned}$$

$$\left[\begin{aligned} \because \sin A - \sin B &= 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \\ \&\cos A + \cos B &= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \end{aligned} \right]$$

Question 21:

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$$

Solution:

$$\begin{aligned} LHS &= \frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} \\ &= \frac{[\cos 4x + \cos 2x] + \cos 3x}{[\sin 4x + \sin 2x] + \sin 3x} \\ &= \frac{\left[2 \cos \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) \right] + \cos 3x}{\left[2 \sin \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) \right] + \sin 3x} \\ &= \frac{2 \cos 3x \cos x + \cos 3x}{2 \sin 3x \cos x + \sin 3x} \\ &= \frac{\cos 3x (2 \cos x + 1)}{\sin 3x (2 \cos x + 1)} \\ &= \frac{\cos 3x}{\sin 3x} \\ &= \cot 3x \\ &= RHS \end{aligned}$$

$$\left[\begin{aligned} \because \cos A + \cos B &= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\ \&\sin A + \sin B &= 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \end{aligned} \right]$$

Question 22:

$$\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$$

Solution:

$$LHS = \cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x$$

$$= \cot x \cot 2x - \cot 3x (\cot 2x + \cot x)$$

$$= \cot x \cot 2x - \cot (2x + x) (\cot 2x + \cot x)$$

$$= \cot x \cot 2x - \left[\frac{\cot 2x \cot x - 1}{\cot 2x + \cot x} \right] (\cot 2x + \cot x) \quad \left[\because \cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} \right]$$

$$= \cot x \cot 2x - [\cot 2x \cot x - 1]$$

$$= \cot x \cot 2x - \cot x \cot 2x + 1$$

$$= 1$$

$$= RHS$$

Question 23:

$$\tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$$

Solution:

$$LHS = \tan 4x$$

$$= \tan 2(2x)$$

$$= \frac{2 \tan 2x}{1 - \tan^2 2x}$$

$$= \frac{2 \left(\frac{2 \tan x}{1 - \tan^2 x} \right)}{1 - \left(\frac{2 \tan x}{1 - \tan^2 x} \right)^2}$$

$$= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{1 - \left(\frac{4 \tan^2 x}{1 + \tan^4 x - 2 \tan^2 x} \right)}$$

$$= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{\left(\frac{1 + \tan^4 x - 2 \tan^2 x - 4 \tan^2 x}{1 + \tan^4 x - 2 \tan^2 x} \right)}$$

$$= \left(\frac{4 \tan x}{1 - \tan^2 x} \right) \times \left(\frac{1 + \tan^4 x - 2 \tan^2 x}{1 + \tan^4 x - 6 \tan^2 x} \right)$$

$$= \frac{4 \tan x (1 - \tan^2 x)^2}{(1 - \tan^2 x)(1 + \tan^4 x - 6 \tan^2 x)}$$

$$= \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$$

$$= RHS$$

$$\left[\because \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \right]$$

$$\left[\because \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \right]$$

$$\left[\because (a-b)^2 = a^2 + b^2 - 2ab \right]$$

$$\left[\because a^2 + b^2 - 2ab = (a-b)^2 \right]$$

Question 24:

$$\cos 4x = 1 - 8 \sin^2 x \cos^2 x$$

Solution:

$$LHS = \cos 4x$$

$$= \cos 2(2x)$$

$$= 1 - 2 \sin^2 2x$$

$$= 1 - 2(2 \sin x \cos x)^2$$

$$= 1 - 2(4 \sin^2 x \cos^2 x)$$

$$= 1 - 8 \sin^2 x \cos^2 x$$

$$= RHS$$

$$\left[\because \cos 2A = 1 - 2 \sin^2 x \right]$$

$$\left[\because \sin 2A = 2 \sin x \cos x \right]$$

Question 25:

$$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

Solution:

$$LHS = \cos 6x$$

$$= \cos 3(2x)$$

$$= 4 \cos^3 2x - 3 \cos 2x$$

$$= 4(2 \cos^2 x - 1)^3 - 3(2 \cos^2 x - 1)$$

$$= 4(8 \cos^6 x - 12 \cos^4 x + 6 \cos^2 x - 1) - 6 \cos^2 x + 3$$

$$= 32 \cos^6 x - 48 \cos^4 x + 24 \cos^2 x - 6 \cos^2 x + 3$$

$$= 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

$$= RHS$$

$$[\because \cos 3A = 4 \cos^3 x - 3 \cos x]$$

$$[\because \cos 2A = 2 \cos^2 x - 1]$$

$$[\because (a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2]$$

Gyanai

EXERCISE 3.4

Find the principal and general solutions of the following equations:

Question 1:

$$\tan x = \sqrt{3}$$

Solution:

It is given that $\tan x = \sqrt{3}$

We know that $\tan \frac{\pi}{3} = \sqrt{3}$ and

$$\begin{aligned}\tan \frac{4\pi}{3} &= \tan \left(\pi + \frac{\pi}{3} \right) \\ &= \tan \frac{\pi}{3} \\ &= \sqrt{3}\end{aligned}$$

Hence the principal solutions are $x = \frac{\pi}{3}$ and $\frac{4\pi}{3}$.

Now,

$$\tan x = \tan \frac{\pi}{3}$$

Therefore, $x = n\pi + \frac{\pi}{3}$, where $n \in \mathbb{Z}$.

Hence the general solution is $x = n\pi + \frac{\pi}{3}$, where $n \in \mathbb{Z}$.

Question 2:

$$\sec x = 2$$

Solution:

It is given that $\sec x = 2$

We know that $\sec \frac{\pi}{3} = 2$ and

$$\begin{aligned}\sec \frac{5\pi}{3} &= \sec \left(2\pi - \frac{\pi}{3} \right) \\ &= \sec \frac{\pi}{3} \\ &= 2\end{aligned}$$

Hence the principal solutions are $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

Now,

$$\sec x = \sec \frac{\pi}{3}$$

$$\cos x = \cos \frac{\pi}{3}$$

Therefore, $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

Hence the general solution is $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

Question 3:

$$\cot x = -\sqrt{3}$$

Solution:

It is given that $\cot x = -\sqrt{3}$

We know that $\cot \frac{\pi}{6} = \sqrt{3}$

Therefore,

$$\cot\left(\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6}$$

$$\cot\left(\frac{5\pi}{6}\right) = -\sqrt{3}$$

and

$$\cot\left(2\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6}$$

$$\cot\left(\frac{11\pi}{6}\right) = -\sqrt{3}$$

Hence the principal solutions are $x = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

Now,

$$\cot x = \cot\left(\frac{5\pi}{6}\right)$$

$$\tan x = \tan\left(\frac{5\pi}{6}\right)$$

Therefore, $x = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

Hence the general solution is $x = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

Question 4:

$$\operatorname{cosec} x = -2$$

Solution:

It is given that $\operatorname{cosec} x = -2$

We know that $\operatorname{cosec} \frac{\pi}{6} = 2$ and

Therefore,

$$\operatorname{cosec} \left(\pi + \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6}$$

$$\operatorname{cosec} \left(\frac{7\pi}{6} \right) = -2$$

and

$$\operatorname{cosec} \left(2\pi - \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6}$$

$$\operatorname{cosec} \left(\frac{11\pi}{6} \right) = -2$$

Hence the principal solutions are $x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Now,

$$\operatorname{cosec} x = \operatorname{cosec} \frac{7\pi}{6}$$

$$\sin x = \sin \frac{7\pi}{6}$$

Therefore, $x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in Z$.

Hence the general solution is $x = n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in Z$.

Find the general solution for each of the following equations:

Question 5:

$$\cos 4x = \cos 2x$$

Solution:

$$\cos 4x - \cos 2x = 0$$

$$-2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right) = 0$$

$$\left[\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \right]$$

$$\sin 3x \sin x = 0$$

$$\sin 3x = 0 \quad \text{or} \quad \sin x = 0$$

$$3x = n\pi \quad \text{or} \quad x = n\pi \quad \text{where } n \in \mathbb{Z}$$

$$x = \frac{n\pi}{3} \quad \text{or} \quad x = n\pi$$

Therefore, $x = \frac{n\pi}{3}$ or $n\pi$, where $n \in \mathbb{Z}$

Question 6:

$$\cos 3x + \cos x - \cos 2x = 0$$

Solution:

$$\cos 3x + \cos x - \cos 2x = 0$$

$$2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) - \cos 2x = 0 \quad \left[\because \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right]$$

$$2\cos 2x \cos x - \cos 2x = 0$$

$$\cos 2x (2\cos x - 1) = 0$$

$$\cos 2x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$

$$2x = (2n+1)\frac{\pi}{2} \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$x = (2n+1)\frac{\pi}{4} \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3}$$

Therefore, $x = (2n+1)\frac{\pi}{4}$ or $\left(2n\pi \pm \frac{\pi}{3}\right)$, where $n \in \mathbb{Z}$

Question 7:

$$\sin 2x + \cos x = 0$$

Solution:

$$\sin 2x + \cos x = 0$$

$$2 \sin x \cos x + \cos x = 0 \quad \left[\because \sin 2x = 2 \sin x \cos x \right]$$

$$\cos x (2 \sin x + 1) = 0$$

Now,

$$\cos x = 0 \quad \text{or} \quad 2 \sin x + 1 = 0$$

Therefore,

$$\cos x = 0$$

$$x = (2n+1) \frac{\pi}{2} \quad n \in Z$$

Or,

$$2 \sin x + 1 = 0$$

$$\sin x = -\frac{1}{2}$$

$$= -\sin \frac{\pi}{6}$$

$$= \sin \left(\pi + \frac{\pi}{6} \right)$$

$$\sin x = \sin \frac{7\pi}{6}$$

$$x = n\pi + (-1)^n \frac{7\pi}{6} \quad n \in Z$$

Therefore, $x = (2n+1) \frac{\pi}{2}$ or $n\pi + (-1)^n \frac{7\pi}{6}$, where $n \in Z$

Question 8:

$$\sec^2 2x = 1 - \tan 2x$$

Solution:

$$\sec^2 2x = 1 - \tan 2x$$

$$1 - \tan^2 2x = 1 - \tan 2x$$

$$\tan^2 2x - \tan 2x = 0$$

$$\tan 2x (\tan 2x - 1) = 0$$

Now,

$$\tan 2x = 0 \quad \text{or} \quad \tan 2x - 1 = 0$$

Therefore,

$$\tan 2x = 0$$

$$2x = n\pi \quad n \in Z$$

$$x = \frac{n\pi}{2}$$

Or,

$$\tan 2x - 1 = 0$$

$$\tan 2x = 1$$

$$= -\tan \frac{\pi}{4}$$

$$= \tan \left(\pi - \frac{\pi}{4} \right)$$

$$= \tan \frac{3\pi}{4}$$

$$2x = n\pi + \frac{3\pi}{4} \quad n \in Z$$

$$x = \frac{n\pi}{2} + \frac{3\pi}{8}$$

Therefore, $x = \frac{n\pi}{2}$ or $\left(\frac{n\pi}{2} + \frac{3\pi}{8} \right)$, where $n \in Z$

Question 9:

$$\sin x + \sin 3x + \sin 5x = 0$$

Solution:

$$\sin x + \sin 3x + \sin 5x = 0$$

$$2 \sin \left(\frac{5x+x}{2} \right) \cos \left(\frac{5x-x}{2} \right) + \sin 3x = 0$$

$$2 \sin 3x \cos 2x + \sin 3x = 0$$

$$\sin 3x (2 \cos 2x + 1) = 0$$

$$\left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

Now,

$$\sin 3x = 0 \quad \text{or} \quad 2 \cos 2x + 1 = 0$$

Therefore,

$$\sin 3x = 0$$

$$3x = n\pi \quad n \in Z$$

$$x = \frac{n\pi}{3}$$

Or,

$$2\cos 2x + 1 = 0$$

$$\cos 2x = -\frac{1}{2}$$

$$= -\cos \frac{\pi}{3}$$

$$= \cos \left(\pi - \frac{\pi}{3} \right)$$

$$= \cos \frac{2\pi}{3}$$

$$2x = 2n\pi \pm \frac{2\pi}{3} \quad n \in Z$$

$$x = n\pi \pm \frac{\pi}{3}$$

Therefore, $x = \frac{n\pi}{3}$ or $\left(n\pi \pm \frac{\pi}{3} \right)$, where $n \in Z$

MISCELLANEOUS EXERCISE

Prove that:

Question 1:

$$2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$$

Solution:

$$\begin{aligned} LHS &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \left(\frac{\frac{3\pi}{13} + \frac{5\pi}{13}}{2} \right) \cos \left(\frac{\frac{3\pi}{13} - \frac{5\pi}{13}}{2} \right) \left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \left(-\frac{\pi}{13} \right) \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \frac{\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \left[\cos \frac{9\pi}{13} + \cos \frac{4\pi}{13} \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \left(\frac{\frac{9\pi}{13} + \frac{4\pi}{13}}{2} \right) \cos \left(\frac{\frac{9\pi}{13} - \frac{4\pi}{13}}{2} \right) \right] \left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \frac{\pi}{2} \cos \frac{5\pi}{26} \right] \\ &= 2 \cos \frac{\pi}{13} \times 2 \times 0 \times \cos \frac{5\pi}{26} \left[\because \cos \frac{\pi}{2} = 0 \right] \\ &= 0 \\ &= RHS \end{aligned}$$

Question 2:

$$(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$$

Solution:

$$\begin{aligned}
LHS &= (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x \\
&= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x \\
&= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x) \\
&= \cos(3x - x) - \cos 2x & \left[\begin{array}{l} \because \cos(A - B) = \cos A \cos B + \sin A \sin B \\ \& \cos 2A = \cos^2 A - \sin^2 A \end{array} \right] \\
&= \cos 2x - \cos 2x \\
&= 0 \\
&= RHS
\end{aligned}$$

Question 3:

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}$$

Solution:

$$\begin{aligned}
LHS &= (\cos x + \cos y)^2 + (\sin x - \sin y)^2 \\
&= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y & \left[\begin{array}{l} \because (a+b)^2 = a^2 + b^2 + 2ab \\ \& (a-b)^2 = a^2 + b^2 - 2ab \end{array} \right] \\
&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y) \\
&= 1 + 1 + 2 \cos(x+y) & \left[\begin{array}{l} \because (\cos^2 A + \sin^2 A) = 1 \\ \& \cos(A+B) = \cos A \cos B - \sin A \sin B \end{array} \right] \\
&= 2 + 2 \cos(x+y) \\
&= 2 \left[1 + \cos 2 \left(\frac{x+y}{2} \right) \right] \\
&= 2 \left[1 + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 \right] & [\because \cos 2A = 2 \cos^2 A - 1] \\
&= 4 \cos^2 \frac{x+y}{2} \\
&= RHS
\end{aligned}$$

Question 4:

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$$

Solution:

$$\begin{aligned}
LHS &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\
&= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y - 2\sin x \sin y \quad \left[\begin{array}{l} \because (a+b)^2 = a^2 + b^2 + 2ab \\ \& (a-b)^2 = a^2 + b^2 - 2ab \end{array} \right] \\
&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2(\cos x \cos y + \sin x \sin y) \\
&= 1 + 1 - 2\cos(x-y) \quad \left[\begin{array}{l} \because (\cos^2 A + \sin^2 A) = 1 \\ \& \cos(A-B) = \cos A \cos B + \sin A \sin B \end{array} \right] \\
&= 2 - 2\cos(x-y) \\
&= 2 \left[1 - \cos 2\left(\frac{x-y}{2}\right) \right] \\
&= 2 \left[1 - \left\{ 1 - 2\sin^2\left(\frac{x-y}{2}\right) \right\} \right] \quad \left[\because \cos 2A = 2\cos^2 A - 1 \right] \\
&= 2 \left[1 - 1 + 2\sin^2\left(\frac{x-y}{2}\right) \right] \\
&= 4\sin^2 \frac{x-y}{2} \\
&= RHS
\end{aligned}$$

Question 5:

$$\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$$

Solution:

$$\begin{aligned}
LHS &= \sin x + \sin 3x + \sin 5x + \sin 7x \\
&= (\sin 5x + \sin x) + (\sin 7x + \sin 3x) \\
&= \left[2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) \right] + \left[2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right) \right] \\
&\quad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right] \\
&= 2\sin 3x \cos 2x + 2\sin 5x \cos 2x \\
&= 2\cos 2x (\sin 5x + \sin 3x) \\
&= 2\cos 2x \left[2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right) \right] \quad \left[\because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right] \\
&= 2\cos 2x [2\sin 4x \cos x] \\
&= 4\cos x \cos 2x \sin 4x \\
&= RHS
\end{aligned}$$

Question 6:

$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

Solution:

$$LHS = \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$$

$$= \frac{2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) + 2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) + 2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}$$

$$\left[\begin{array}{l} \because \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \& \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{array} \right]$$

$$= \frac{2\sin 6x \cos x + 2\sin 6x \cos 3x}{2\cos 6x \cos x + 2\cos 6x \cos 3x}$$

$$= \frac{2\sin 6x(\cos x + \cos 3x)}{2\cos 6x(\cos x + \cos 3x)}$$

$$= \frac{\sin 6x}{\cos 6x}$$

$$= \tan 6x$$

$$= RHS$$

Question 7:

$$\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$$

Solution:

$$LHS = \sin 3x + \sin 2x - \sin x$$

$$= \sin 3x + (\sin 2x - \sin x)$$

$$= \sin 2\left(\frac{3x}{2}\right) + \left[2 \cos\left(\frac{2x+x}{2}\right) \sin\left(\frac{2x-x}{2}\right)\right] \quad \left[\because \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2 \sin \frac{3x}{2} \cos \frac{3x}{2} + 2 \cos \frac{3x}{2} \sin \frac{x}{2} \quad [\because \sin 2A = 2 \sin A \cos A]$$

$$= 2 \cos \frac{3x}{2} \left(\sin \frac{3x}{2} + \sin \frac{x}{2} \right)$$

$$= 2 \cos \frac{3x}{2} \left[2 \sin \left(\frac{\frac{3x}{2} + \frac{x}{2}}{2} \right) \cos \left(\frac{\frac{3x}{2} - \frac{x}{2}}{2} \right) \right] \quad \left[\because \sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2 \cos \frac{3x}{2} \left[2 \sin x \cos \frac{x}{2} \right]$$

$$= 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$$

$$= RHS$$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following:

Question 8:

$\tan x = -\frac{4}{3}$, x in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Therefore,

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\tan x = -\frac{4}{3}$

$$\begin{aligned}
 \sec^2 x &= 1 + \tan^2 x \\
 &= 1 + \left(-\frac{4}{3}\right)^2 \\
 &= 1 + \frac{16}{9} \\
 &= \frac{25}{9}
 \end{aligned}$$

$$\sec x = \pm \sqrt{\frac{25}{9}}$$

$$\frac{1}{\cos x} = \pm \frac{5}{3}$$

$$\cos x = \pm \frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$$\cos x = -\frac{3}{5}$$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{3}{5}$$

$$2\cos^2 \frac{x}{2} - 1 = -\frac{3}{5}$$

$$2\cos^2 \frac{x}{2} = 1 - \frac{3}{5}$$

$$\cos^2 \frac{x}{2} = \frac{2}{5} \times \frac{1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1}{5}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1}{5}}$$

Since, $\cos \frac{x}{2}$ lies in quadrant I and positive, $\cos \frac{x}{2} = \frac{1}{\sqrt{5}}$

Now,

$$\sin^2 \frac{x}{2} = 1 - \cos^2 \frac{x}{2} \quad \left[\because \sin^2 A + \cos^2 A = 1 \right]$$

$$= 1 - \left(\frac{1}{\sqrt{5}} \right)^2$$

$$= 1 - \frac{1}{5}$$

$$= \frac{4}{5}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{4}{5}}$$

$$\sin \frac{x}{2} = \pm \frac{2}{\sqrt{5}}$$

Since, $\sin \frac{x}{2}$ lies in quadrant I and positive, $\sin \frac{x}{2} = \frac{2}{\sqrt{5}}$

Now,

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}$$

$$= \frac{\left(\frac{2}{\sqrt{5}} \right)}{\left(\frac{1}{\sqrt{5}} \right)}$$

$$= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{1}$$

$$= 2$$

Therefore, $\sin \frac{x}{2} = \frac{2}{\sqrt{5}}$, $\cos \frac{x}{2} = \frac{1}{\sqrt{5}}$ and $\tan \frac{x}{2} = 2$.

Question 9:

$\cos x = -\frac{1}{3}$, x in quadrant III.

Solution:

Since x lies in quadrant III

$$\pi < x < \frac{3\pi}{2}$$

Therefore,

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Hence, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are negative while $\sin \frac{x}{2}$ is positive as all lie in quadrant II.

It is given that $\cos x = -\frac{1}{3}$

$$\cos 2\left(\frac{x}{2}\right) = -\frac{1}{3}$$

$$2\cos^2 \frac{x}{2} - 1 = -\frac{1}{3}$$

$$2\cos^2 \frac{x}{2} = 1 - \frac{1}{3}$$

$$\cos^2 \frac{x}{2} = \frac{2}{3} \times \frac{1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1}{3}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1}{3}}$$

Since, $\cos \frac{x}{2}$ is negative

So,

$$\cos \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

Now,

$$\sin^2 \frac{x}{2} = 1 - \cos^2 \frac{x}{2} \quad \left[\because \sin^2 A + \cos^2 A = 1 \right]$$

$$= 1 - \left(-\frac{\sqrt{3}}{3} \right)^2$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{2}{3}}$$

Since, $\sin \frac{x}{2}$ positive,

So,

$$\begin{aligned}\sin \frac{x}{2} &= \sqrt{\frac{2}{3}} \\ &= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{6}}{3}\end{aligned}$$

Now,

$$\begin{aligned}\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \\ &= \frac{\frac{\sqrt{6}}{3}}{\left(-\frac{\sqrt{3}}{3}\right)} \\ &= \frac{\sqrt{6}}{3} \times \left(-\frac{3}{\sqrt{3}}\right) \\ &= -\sqrt{2}\end{aligned}$$

Therefore, $\sin \frac{x}{2} = \frac{\sqrt{6}}{3}$, $\cos \frac{x}{2} = -\frac{\sqrt{3}}{3}$ and $\tan \frac{x}{2} = -\sqrt{2}$.

Question 10:

$\sin x = \frac{1}{4}$, x in quadrant II.

Solution:

Since x lies in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Therefore,

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ lie in quadrant I and all are positive.

It is given that $\sin x = \frac{1}{4}$

Therefore,

$$\begin{aligned}
 \cos^2 x &= 1 - \sin^2 x & [\because \sin^2 A + \cos^2 A = 1] \\
 &= 1 - \left(\frac{1}{4}\right)^2 \\
 &= 1 - \frac{1}{16} \\
 &= \frac{15}{16} \\
 \cos x &= \pm \sqrt{\frac{15}{16}}
 \end{aligned}$$

Since, $\cos x$ lies in quadrant II and negative
So,

$$\begin{aligned}
 \cos x &= -\sqrt{\frac{15}{16}} \\
 \cos 2\left(\frac{x}{2}\right) &= -\frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} - 1 &= -\frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} &= 1 - \frac{\sqrt{15}}{4} \\
 2\cos^2 \frac{x}{2} &= \frac{4 - \sqrt{15}}{4} \\
 \cos^2 \frac{x}{2} &= \frac{4 - \sqrt{15}}{8} \\
 \cos \frac{x}{2} &= \pm \sqrt{\frac{4 - \sqrt{15}}{8}}
 \end{aligned}$$

Since, $\cos \frac{x}{2}$ lies in quadrant I and is positive.

So,

$$\begin{aligned}
 \cos \frac{x}{2} &= \sqrt{\frac{4 - \sqrt{15}}{8}} \\
 &= \sqrt{\frac{4 - \sqrt{15}}{8} \times \frac{2}{2}} \\
 &= \frac{\sqrt{8 - 2\sqrt{15}}}{4}
 \end{aligned}$$

Now,

$$\sin^2 \frac{x}{2} = 1 - \cos^2 \frac{x}{2} \quad \left[\because \sin^2 A + \cos^2 A = 1 \right]$$

$$= 1 - \left(\frac{\sqrt{8 - 2\sqrt{15}}}{4} \right)^2$$

$$= 1 - \frac{8 - 2\sqrt{15}}{16}$$

$$= \frac{16 - 8 + 2\sqrt{15}}{16}$$

$$= \frac{8 + 2\sqrt{15}}{16}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{8 + 2\sqrt{15}}{16}}$$

Since, $\sin \frac{x}{2}$ positive,

So,

$$\sin \frac{x}{2} = \frac{\sqrt{8 + 2\sqrt{15}}}{4}$$

Now,

$$\begin{aligned}\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \\&= \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4} \right)} \\&= \left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right) \times \left(\frac{4}{\sqrt{8-2\sqrt{15}}} \right) \\&= \sqrt{\frac{2(4+\sqrt{15})}{2(4-\sqrt{15})}} \\&= \sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}} \times \frac{4+\sqrt{15}}{4+\sqrt{15}}} \\&= \sqrt{\frac{(4+\sqrt{15})^2}{16-15}} \\&= 4+\sqrt{15}\end{aligned}$$

Therefore, $\sin \frac{x}{2} = \frac{\sqrt{8+2\sqrt{15}}}{4}$, $\cos \frac{x}{2} = \frac{\sqrt{8-2\sqrt{15}}}{4}$ and $\tan \frac{x}{2} = 4+\sqrt{15}$.